

数字信号处理

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第三章

离散傅里叶变换

§ 3-3 离散傅里叶级数(DFS)

一、DFS变换的推导

由DTFT推导DFT

由DTFT
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

Q $X(e^{j\omega}) = X(e^{j(\omega+2p)}) \quad \therefore \text{令 } \tilde{X}(e^{j\omega}) \triangleq X(e^{j\omega})$

假定 $x(n) = 0$, 当 $n < 0$, $n > N-1$ (有限长)

$$\tilde{X}(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

$$\tilde{X}(k) \triangleq X(e^{j\omega}) \Big|_{\omega=\frac{2p}{N}k} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2p}{N}kn} = \tilde{X}(k+N) \quad (3-11)$$
$$0 \leq k \leq N-1$$

采样, 周期性离散频率函数

时域序列周期化

§ 3-3 离散傅里叶级数(DFS)

假定 $\tilde{x}(n+N) = \tilde{x}(n)$, $N \sim$ 周期

$$\tilde{x}(n) = x(n), \quad 0 \leq n \leq N-1$$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2p}{N}kn} \quad (3-11)$$

可见 $\tilde{x}(n) \longrightarrow \tilde{X}(k)$

问题 $\tilde{X}(k) \xrightarrow{?} \tilde{x}(n)$

$$\text{令 } \tilde{x}'(n) \triangleq \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2p}{N}kn}$$

代入 (3-11) 式

$$\begin{aligned} &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}(m) e^{-j\frac{2p}{N}km} \right) e^{j\frac{2p}{N}kn} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \tilde{x}(m) \sum_{k=0}^{N-1} e^{j\frac{2p}{N}k(n-m)} \end{aligned}$$

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$$\tilde{x}'(n) \triangleq \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{m=0}^{N-1} \tilde{x}(m) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)}$$

可以证明 $\frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(n-m)} = \begin{cases} 1 & n = m + Nl \\ 0 & n \neq m + Nl \end{cases}$ 正交定理

$$\therefore \tilde{x}'(n) \triangleq \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j\frac{2\pi}{N}kn} = \left[\sum_{m=0}^{N-1} \tilde{x}(m) \right]_{n=m} = \tilde{x}(n)$$

$$\therefore \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}kn} \quad (3-13)$$

$$\tilde{X}(k) \longrightarrow \tilde{x}(n)$$

§ 3-3 离散傅里叶级数(DFS)

结合 (3-11)、(3-13) 式,

$$\tilde{x}(n) \xleftrightarrow{DFS} \tilde{X}(k)$$

为方便起见, 令

$$W_N \triangleq e^{-j\frac{2\pi}{N}} \longrightarrow W_N \text{因子}$$

DFS变换:

$$\tilde{X}(k) \triangleq DFS[\tilde{x}(n)] = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn}, \quad \forall k$$

$$\tilde{x}(n) \triangleq IDFS[\tilde{X}(k)] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn}, \quad \forall n$$

§ 3-3 离散傅里叶级数(DFS)

DFS例题：习题集P37 1

已知 $x(n) = \{14 \ 12 \ 10 \ 8 \ 6 \ 10\}$ ，求DFS

$$\text{解： } \tilde{X}(k) = \text{DFS}[x(n)] = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad \forall k$$

$$= \sum_{n=0}^5 x(n)W_6^{kn} = \sum_{n=0}^5 x(n)e^{-j\frac{2\pi}{6}nk}$$

$$\tilde{X}(0) = 60 \qquad \tilde{X}(3) = 0$$

$$\tilde{X}(1) = 9 - j3\sqrt{3} \qquad \tilde{X}(4) = 3 - j\sqrt{3}$$

$$\tilde{X}(2) = 3 + j\sqrt{3} \qquad \tilde{X}(5) = 9 + j3\sqrt{3}$$

§ 3-3 离散傅里叶级数(DFS)

二、DFS的主要性质

1. 线性特性

迭加原理

$$\tilde{x}_3(n) = a\tilde{x}_1(n) + b\tilde{x}_2(n)$$

$$\tilde{X}_3(k) = DFS[a\tilde{x}_1(n) + b\tilde{x}_2(n)] = a\tilde{X}_1(k) + b\tilde{X}_2(k)$$

2. 移位特性

(1) 时域移位

$$\text{若 } x(n) \xleftrightarrow{DFS} X(k), \text{ 则 } x(n-m) \xleftrightarrow{DFS} W_N^{mk} X(k)$$

(2) 频域移位

$$\text{若 } X(k) \xleftrightarrow{IDFS} x(n), \text{ 则 } X(k-l) \xleftrightarrow{IDFS} W_N^{-nl} x(n)$$

§ 3-3 离散傅里叶级数(DFS)

3. 周期卷积特性

(1) 时域

$$\forall \tilde{x}_1(n) \xleftrightarrow{DFS} \tilde{X}_1(k), \quad \tilde{x}_2(n) \xleftrightarrow{DFS} \tilde{X}_2(k)$$

$$\tilde{X}(k) = \tilde{X}_1(k) \tilde{X}_2(k)$$

IDFS $\updownarrow$

$$\begin{aligned} x(n) &= \sum_{m=0}^{N-1} \tilde{x}_1(m) \tilde{x}_2(n-m) \\ &= \sum_{m=0}^{N-1} \tilde{x}_2(m) \tilde{x}_1(n-m) \end{aligned}$$

$$= \tilde{x}_1(n) \tilde{\otimes} \tilde{x}_2(n) \rightarrow \text{周期卷积}$$

比较:

$$* \quad \sum_{m=-\infty}^{+\infty}$$

$$\tilde{\otimes} \quad \sum_{m=0}^{N-1} \quad \text{仅一个周期}$$

$$\tilde{x}(n) = \tilde{x}(n+N)$$

时域周期卷积 $\longleftrightarrow$ 频域相乘

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3. 周期卷积特性

(2) 频域

$$x(n) = x_1(n)x_2(n) \xleftrightarrow{DFS} X(k) = \frac{1}{N} X_1(k) \tilde{\otimes} X_2(k)$$

时域相乘 $\longleftrightarrow$ 频域周期卷积

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周期卷积计算例题 课本P75

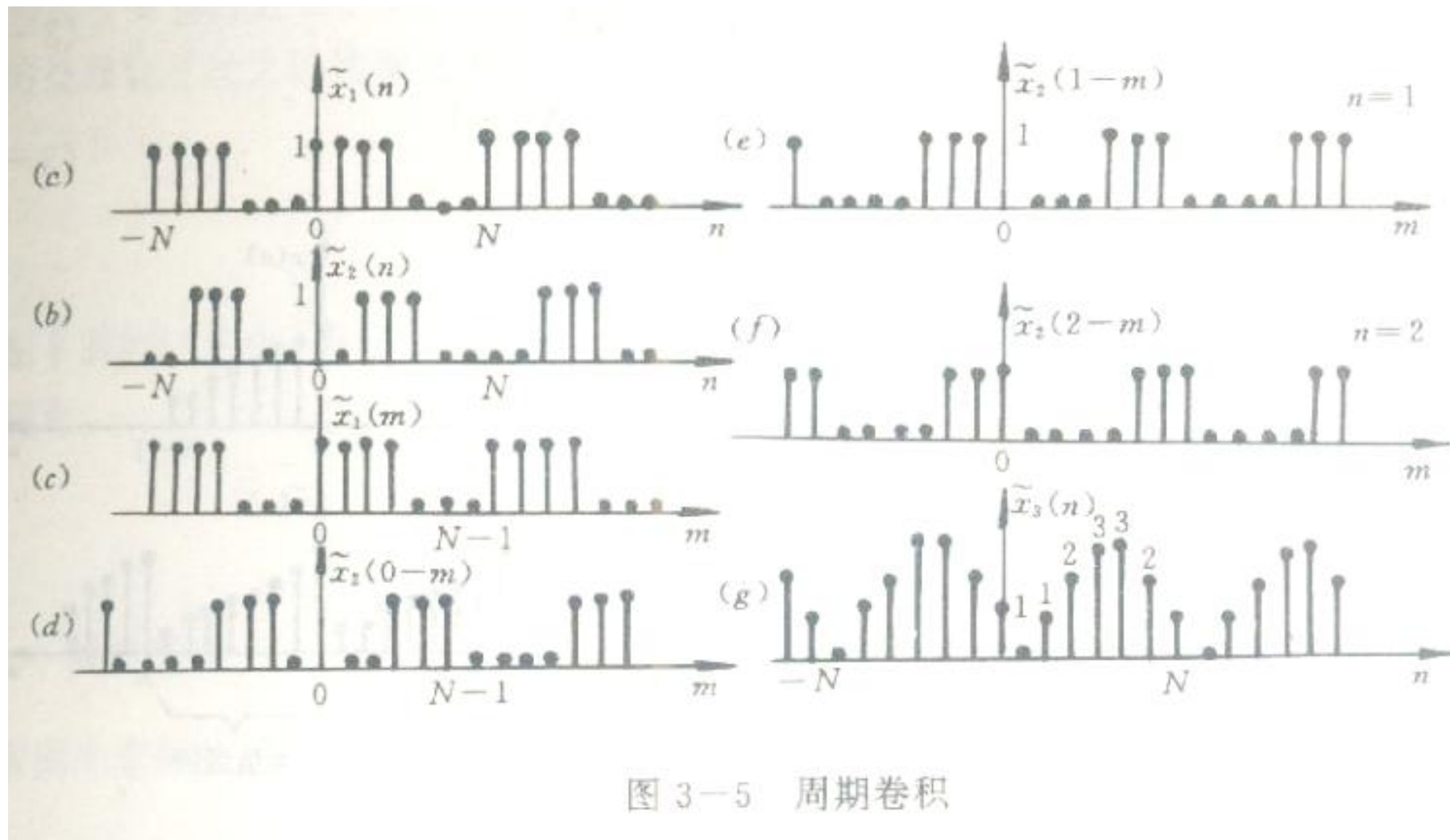


图 3-5 周期卷积

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周期卷积计算例题 习题集P38 3

$$x(n) = \begin{cases} n+1 & 0 \leq n \leq 4 \\ 0 & \text{其他} \end{cases}, \text{ 求 } h(n) = R_4(n-2)$$

$$\text{令 } \tilde{x}(n) = x((n))_6, \tilde{h}(n) = h((n))_6$$

求 $\tilde{x}(n)$ 和 $\tilde{h}(n)$ 的周期卷积

$$\text{解: } \tilde{y}(n) = \tilde{x}(n) \tilde{\otimes} \tilde{h}(n) = \sum_m x(m)h(n-m)$$

$$= \{14 \quad 12 \quad 10 \quad 8 \quad 6 \quad 10\}$$

表 3-3

$\tilde{h}(n-m) \backslash \tilde{x}(m)$ n	1	2	3	4	5	0	$\tilde{y}(n)$
0	0	1	1	1	1	0	14
1	0	0	1	1	1	1	12
2	1	0	0	1	1	1	10
3	1	1	0	0	1	1	8
4	1	1	1	0	0	1	6
5	1	1	1	1	0	0	10

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4. 对称特性

$$(1) \quad \forall \tilde{x}(n) \xleftrightarrow{DFS} \tilde{X}(k)$$

$$\text{则 } \tilde{x}^*(n) \xleftrightarrow{DFS} \tilde{X}^*(-k)$$

$$\tilde{x}^*(-n) \xleftrightarrow{DFS} \tilde{X}^*(k)$$

$$DTFT[x(n)] = X(e^{j\omega})$$

$$DTFT[x^*(n)] = X^*(e^{-j\omega})$$

$$DTFT[x^*(-n)] = X^*(e^{j\omega})$$

$$x(n) \leftrightarrow X(e^{j\omega})$$

$$x^*(n) \leftrightarrow X^*(e^{-j\omega})$$

$$x^*(-n) \leftrightarrow X^*(e^{j\omega})$$

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$$(2) \quad \forall \tilde{x}(n) \xleftrightarrow{DFS} \tilde{X}(k)$$

则 $\text{Re}\{x(n)\} \xleftrightarrow{DFS} X_e(k) = \frac{1}{2} [X(k) + X^*(-k)]$

$$j \text{Im}\{x(n)\} \xleftrightarrow{DFS} X_o(k) = \frac{1}{2} [X(k) - X^*(-k)]$$

$$x_e(n) = \frac{1}{2} [x(n) + x^*(-n)] \xleftrightarrow{DFS} \text{Re}\{X(k)\} \quad \text{共轭对称}$$

$$x_o(n) = \frac{1}{2} [x(n) - x^*(-n)] \xleftrightarrow{DFS} j \text{Im}\{X(k)\} \quad \text{共轭反对称}$$

$$\text{Re}\{x(n)\} = \frac{1}{2} [x(n) + x^*(n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) + X^*(e^{-j\omega})] = X_e(e^{j\omega})$$

$$j \text{Im}\{x(n)\} = \frac{1}{2} [x(n) - x^*(n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) - X^*(e^{-j\omega})] = X_o(e^{j\omega})$$

$$x_e(n) = \frac{1}{2} [x(n) + x^*(-n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) + X^*(e^{j\omega})] = \text{Re}\{X(e^{j\omega})\}$$

$$x_o(n) = \frac{1}{2} [x(n) - x^*(-n)] \leftrightarrow \frac{1}{2} [X(e^{j\omega}) - X^*(e^{j\omega})] = j \text{Im}\{X(e^{j\omega})\}$$

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(3)

$$\forall \tilde{x}(n) = \tilde{x}^*(n) \quad \text{实序列}$$

$$\begin{array}{c} \downarrow \qquad \downarrow \\ \tilde{X}(k) = \tilde{X}^*(-k) \end{array} \quad \text{共轭对称}$$

$$\begin{array}{c} \downarrow \\ |\tilde{X}(k)| = |\tilde{X}(-k)| \end{array} \quad \text{偶对称}$$

$$\arg[\tilde{X}(k)] = -\arg[\tilde{X}(-k)] \quad \text{奇对称}$$

实序列: $x(n) \leftrightarrow X(e^{j\omega})$

$$1. x(n) = x^*(n) \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$2. \begin{cases} X(e^{j\omega}) = \operatorname{Re}\{X(e^{j\omega})\} + j \operatorname{Im}\{X(e^{j\omega})\} \\ X^*(e^{-j\omega}) = \operatorname{Re}\{X(e^{-j\omega})\} - j \operatorname{Im}\{X(e^{-j\omega})\} \end{cases}$$

$$\Rightarrow \begin{cases} \operatorname{Re}\{X(e^{j\omega})\} = \operatorname{Re}\{X(e^{-j\omega})\} \\ \operatorname{Im}\{X(e^{j\omega})\} = -\operatorname{Im}\{X(e^{-j\omega})\} \end{cases}$$

$X(e^{j\omega})$ 实部是偶函数, 虚部是奇函数

$$3. \text{极坐标形式: } X(e^{j\omega}) = |X(e^{j\omega})| e^{j \arg[X(e^{j\omega})]}$$

幅度是 ω 的偶函数 $|X(e^{j\omega})| = |X(e^{-j\omega})|$

相位是 ω 的奇函数 $\arg[X(e^{j\omega})] = -\arg[X(e^{-j\omega})]$